



Finite element method (FEM1)

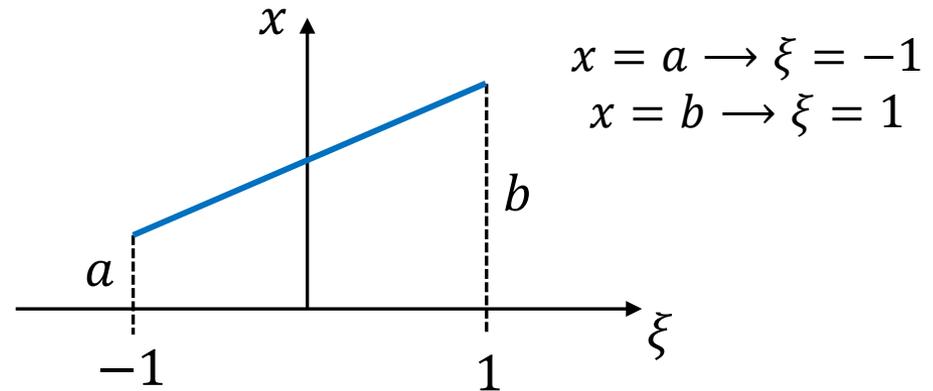
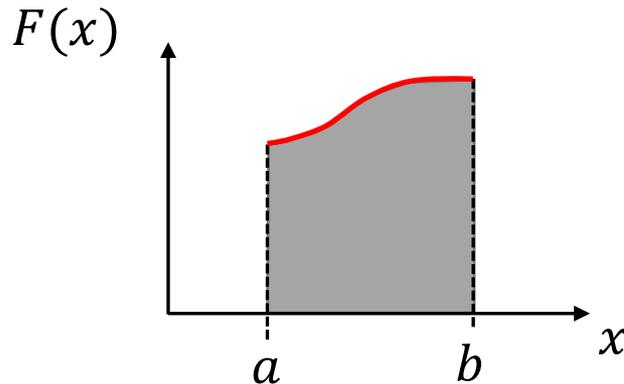
Lecture 5A. Numerical integration

03.2025

Definite integral

x – cartesian coordinate

ξ – natural coordinate



Normalization of a function of one variable $F(x)$:

$$x(\xi) = \frac{b-a}{2} \xi + \frac{a+b}{2} \quad ; \quad f(\xi) = F\left(\frac{b-a}{2} \xi + \frac{a+b}{2}\right) \quad ; \quad dx = \frac{b-a}{2} d\xi$$

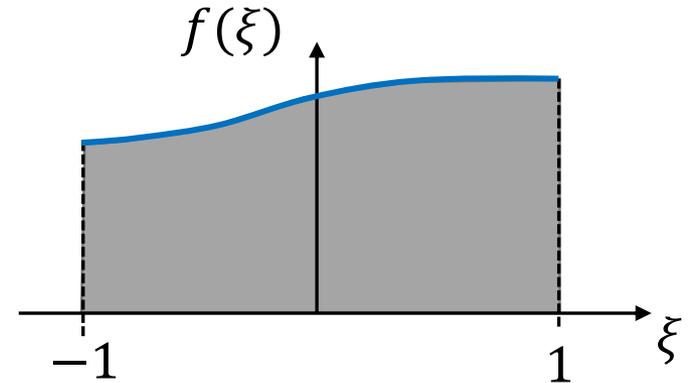
definite integral of the function $F(x)$:

$$\int_a^b F(x) dx = \int_{-1}^1 f(\xi) \frac{b-a}{2} d\xi = \frac{b-a}{2} \int_{-1}^1 f(\xi) d\xi$$

Gaussian quadrature rule

quadrature rule:

$$\int_{-1}^1 f(\xi) d\xi = \sum_{i=1}^n w_i \cdot f(\xi_i) + R_n$$



n – no. of sample points,
 ξ_i – coordinates of sample points
 w_i – weight coefficients
 R_n – rest of the sum

$$R_n = 0 \quad \Rightarrow \quad \frac{d^{2n}f}{d\xi^{2n}} = 0$$

Numerical integration gives the exact value of the integral for polynomials up to $(2n - 1)$ degree.

Gaussian quadrature rule for polynomial functions

$$R_n = 0 \Rightarrow \frac{d^{2n}f}{d\xi^{2n}} = 0$$

for a linear function: $f(\xi) = \alpha \xi + \beta$; $\frac{df}{d\xi} = \alpha$; $\frac{d^2f}{d\xi^2} = 0 \rightarrow 2n = 2 \rightarrow \mathbf{n = 1}$

One point is enough!

$$\int_{-1}^1 (\alpha \xi + \beta) d\xi = w_1 \cdot f(\xi_1) + 0$$

for one Gauss point: $\xi_1 = 0, w_1 = 2 \rightarrow \int_{-1}^1 (\alpha \xi + \beta) d\xi = w_1 \cdot f(0)$

polynomial functions:

2nd order: $f(\xi) = \alpha \xi^2 + \beta \xi + \gamma$; $\frac{df}{d\xi} = 2\alpha \xi + \beta$; $\frac{d^2f}{d\xi^2} = 2\alpha$; $\frac{d^3f}{d\xi^3} = 0 \rightarrow$

$$2n = 3 \rightarrow n = 1.5 \rightarrow \mathbf{n = 2}$$

Two points needed!

for 2 Gauss points: $\xi_1 = -\frac{1}{\sqrt{3}}$; $\xi_2 = \frac{1}{\sqrt{3}}$; $w_1 = w_2 = 1$

$$\rightarrow \int_{-1}^1 (\alpha \xi^2 + \beta \xi + \gamma) d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature rule for polynomial functions

3th order: $f(\xi) = \alpha \xi^3 + \beta \xi^2 + \gamma \xi + \delta$; $\frac{d^4 f}{d\xi^4} = 0 \rightarrow$ **$n = 2$** (2 points)

Two points are enough!

$\rightarrow \int_{-1}^1 (\alpha \xi^3 + \beta \xi^2 + \gamma \xi + \delta) d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

4th order: $f(\xi) = \alpha \xi^4 + \beta \xi^3 + \gamma \xi^2 + \delta \xi + \varphi$; $\frac{d^5 f}{d\xi^5} = 0$

$2n = 5 \rightarrow n = 2.5 \rightarrow$ **$n = 3$**

Three points needed!

for three Gauss points: $\xi_1 = -\sqrt{0.6}$; $\xi_2 = 0$; $\xi_3 = \sqrt{0.6}$;

$w_1 = w_3 = \frac{5}{9}$; $w_2 = \frac{8}{9}$

$$\int_{-1}^1 (\alpha \xi^4 + \beta \xi^3 + \gamma \xi^2 + \delta \xi + \varphi) d\xi =$$

$$= \frac{5}{9} \cdot f(-\sqrt{0.6}) + \frac{8}{9} \cdot f(0) + \frac{5}{9} \cdot f(\sqrt{0.6})$$

Gaussian quadrature rule for polynomial functions

$$\int_{-1}^1 f(\xi) d\xi = \sum_{i=1}^n w_i \cdot f(\xi_i) + R_n$$

$$R_n = 0 \Rightarrow \frac{d^{2n}f}{d\xi^{2n}} = 0$$

Polynomial degree	Number of Gauss points	ξ_i	w_i
1	1	0	2
3	2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9
7	4	-0.861136311594953 -0.339981043584856 +0.339981043584856 +0.861136311594953	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454

The sum of the weight coefficients is always equal 2.
 Numerical integration gives the exact value of the integral for polynomials up to $(2n - 1)$ degree

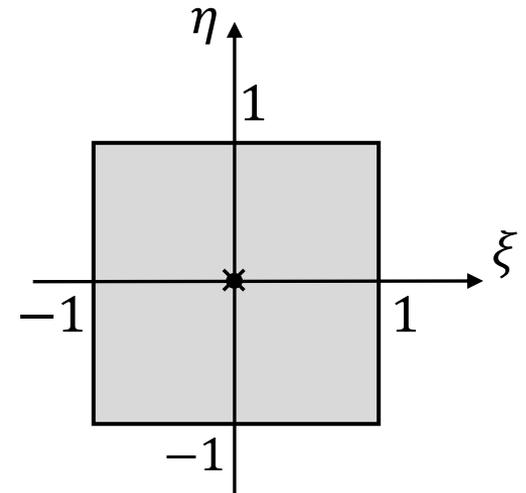
Gaussian quadrature rule for 2D FEs

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \int_{-1}^1 \left(\sum_{i=1}^n (w_i \cdot f(\xi_i, \eta)) \right) d\eta =$$
$$= \sum_{j=1}^n w_j \sum_{i=1}^n (w_i \cdot f(\xi_i, \eta_j)) = \sum_{j=1}^n \sum_{i=1}^n (w_i w_j \cdot f(\xi_i, \eta_j))$$

For one Gaussian point we have:

$$\mathbf{n = 1} : \xi_1 = \eta_1 = 0, \quad w_1 = 2$$

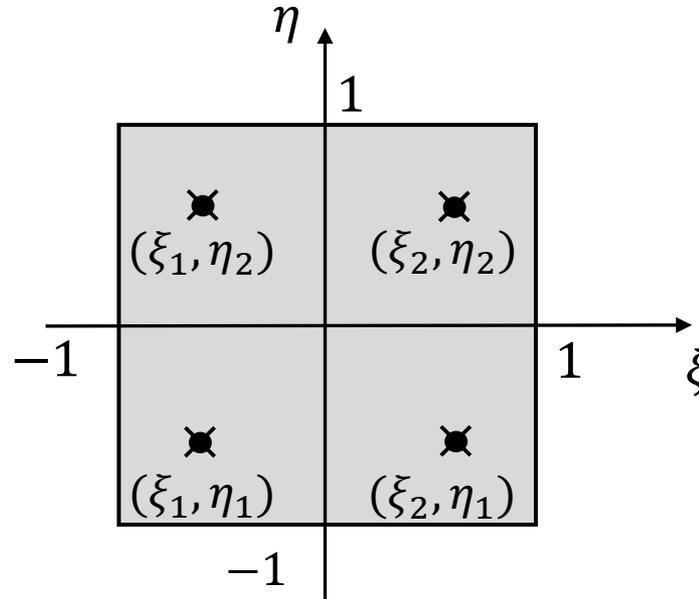
$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = w_1 w_1 \cdot f(0, 0) = 4f(0, 0)$$



Gaussian quadrature rule for 2D FEs

For two Gaussian points
in each direction:

$$n = 2 : \quad \xi_1 = \eta_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \eta_2 = \frac{1}{\sqrt{3}} ; \quad w_1 = w_2 = 1$$



$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta =$$

$$= w_1 w_1 \cdot f(\xi_1, \eta_1) + w_2 w_1 \cdot f(\xi_2, \eta_1) + w_2 w_2 \cdot f(\xi_2, \eta_2) + w_1 w_2 \cdot f(\xi_1, \eta_2) =$$

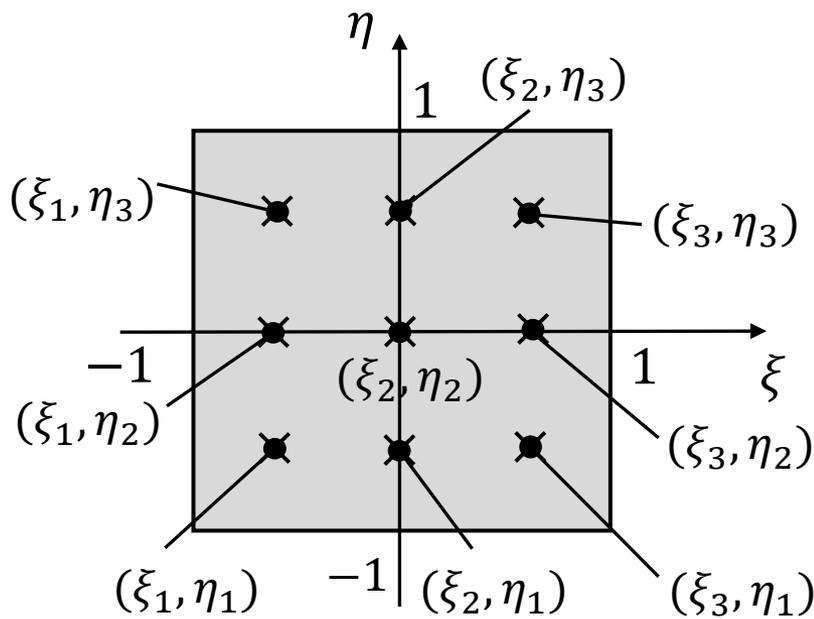
$$= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature rule for 2D FEs

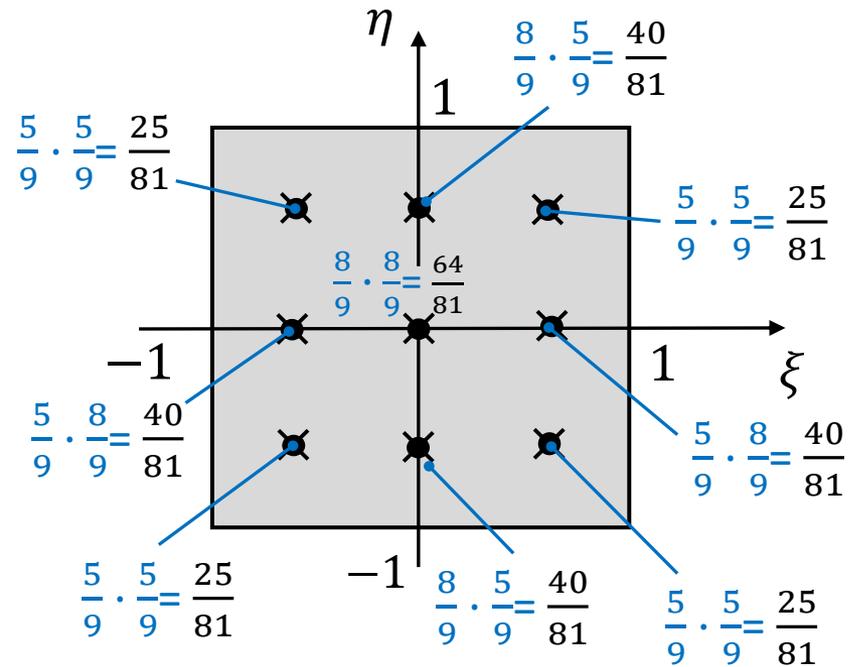
For three Gauss points
in each direction: $n = 3$:

$$\xi_1 = \eta_1 = -\sqrt{0.6}, \quad \xi_2 = \eta_2 = 0, \quad \xi_3 = \eta_3 = \sqrt{0.6}$$

$$w_1 = w_3 = \frac{5}{9} ; w_2 = \frac{8}{9}$$



$$(\xi_i, \eta_j)$$



$$w_i w_j$$

Gaussian quadrature rule for 2D FEs

For three Gauss points
on each direction we have:

$n = 3$:

$$\xi_1 = \eta_1 = -\sqrt{0.6}, \quad \xi_2 = \eta_2 = 0, \quad \xi_3 = \eta_3 = \sqrt{0.6}$$

$$w_1 = w_3 = \frac{5}{9}; \quad w_2 = \frac{8}{9}$$

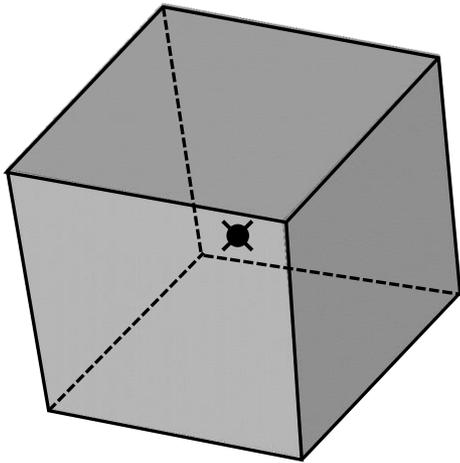
$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta =$$

$$\begin{aligned} &= w_1 w_1 \cdot f(\xi_1, \eta_1) + w_2 w_1 \cdot f(\xi_2, \eta_1) + w_3 w_1 \cdot f(\xi_3, \eta_1) + \\ &+ w_1 w_2 \cdot f(\xi_1, \eta_2) + w_2 w_2 \cdot f(\xi_2, \eta_2) + w_3 w_2 \cdot f(\xi_3, \eta_2) + \\ &+ w_1 w_3 \cdot f(\xi_1, \eta_3) + w_2 w_3 \cdot f(\xi_2, \eta_3) + w_3 w_3 \cdot f(\xi_3, \eta_3) = \end{aligned}$$

$$\begin{aligned} &= \frac{5}{9} \cdot \frac{5}{9} f(-\sqrt{0.6}, -\sqrt{0.6}) + \frac{8}{9} \cdot \frac{5}{9} f(0, -\sqrt{0.6}) + \frac{5}{9} \cdot \frac{5}{9} f(\sqrt{0.6}, -\sqrt{0.6}) + \\ &+ \frac{5}{9} \cdot \frac{8}{9} f(-\sqrt{0.6}, 0) + \frac{8}{9} \cdot \frac{8}{9} f(0, 0) + \frac{5}{9} \cdot \frac{8}{9} f(\sqrt{0.6}, 0) + \\ &+ \frac{5}{9} \cdot \frac{5}{9} f(-\sqrt{0.6}, \sqrt{0.6}) + \frac{8}{9} \cdot \frac{5}{9} f(0, \sqrt{0.6}) + \frac{5}{9} \cdot \frac{5}{9} f(\sqrt{0.6}, \sqrt{0.6}) \end{aligned}$$

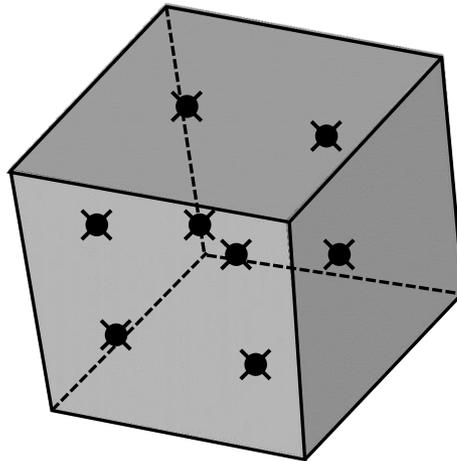
Gaussian quadrature rule for 3D FEs

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{k=1}^n \sum_{j=1}^n \sum_{i=1}^n (w_i w_j w_k \cdot f(\xi_i, \eta_i, \zeta_k))$$



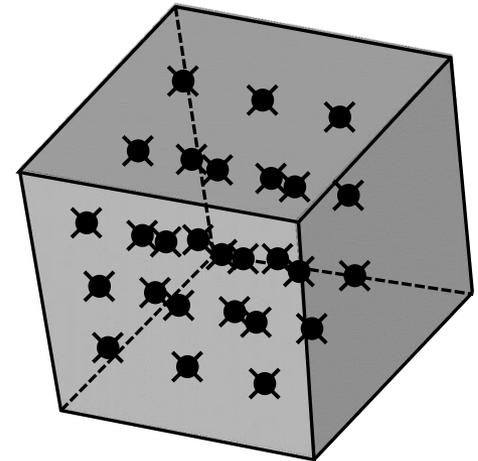
$n = 1$

For one Gaussian point



$n = 2 \quad (2 \times 2 \times 2)$

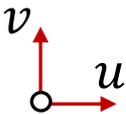
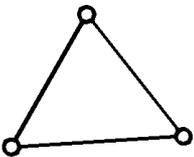
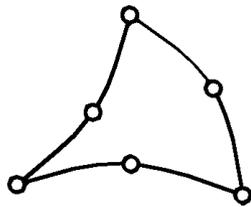
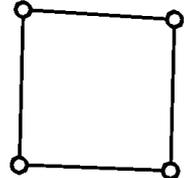
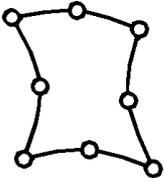
For two Gauss points in each direction



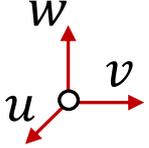
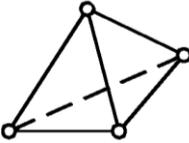
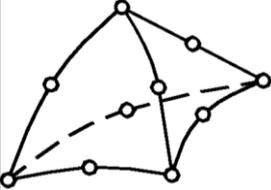
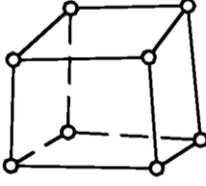
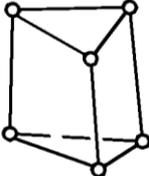
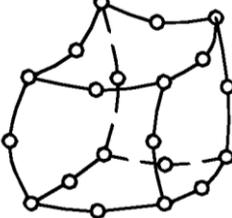
$n = 3 \quad (3 \times 3 \times 3)$

For three Gauss points in each direction

Integration scheme for 2D elements

 2D	3-node	6-node	4-node	8-node
Integration type				
FULL	3	3	2×2	3×3
REDUCED	1	1	1	2×2

Integration scheme for 3D elements

 3D	4-node	10-node	8-node	6-node	20-node
Integration type					
FULL	4	11	$2 \times 2 \times 2$	3×3	$3 \times 3 \times 3$
REDUCED	1	5	1	3×2	$2 \times 2 \times 2$